

# Unsatisfiable hitting clause-sets with three more clauses than variables

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**Abstract.** Hitting clause-sets (as CNFs), known in DNF language as “disjoint” or “orthogonal”, are clause-sets  $F$ , such that any  $C, D \in F$ ,  $C \neq D$ , have a literal  $x \in C$  with  $\bar{x} \in D$ . The set of unsatisfiable such  $F$  is denoted by  $\mathcal{UHIT} \subset \mathcal{MU}$  (minimal unsatisfiability). A basic fact is  $\delta(F) \geq 1$  for  $F \in \mathcal{MU}$ , where the deficiency  $\delta(F) := c(F) - n(F)$  is the difference between the number of clauses and the number of variables. Via the known singular DP-reduction, generalising unit-clause propagation, every  $F \in \mathcal{UHIT}$  can be reduced to its (unique) “non-singular normal form”  $\text{sNF}(F) \in \mathcal{UHIT}'$ , where  $\delta(\text{sNF}(F)) = \delta(F)$ , and  $\mathcal{UHIT}' \subset \mathcal{UHIT}$  is the subset of non-singular elements, i.e., every variable occurs positively as well as negatively at least twice.

The *Finiteness Conjecture* (FC) is that for every  $k \in \mathbb{N}$  the number  $n(F)$  of variables for  $F \in \mathcal{UHIT}'$  with  $\delta(F) = k$  is bounded. This conjecture is part of the project of classifying  $\mathcal{UHIT}_{\delta=k}$ . In this report we prove FC for  $k = 3$  (known for  $k \leq 2$ ). For this, a central novel concept is transferred from number theory (Berger et al 1990 [2]), namely the fundamental notion of *clause-irreducible clause-sets*  $F$ , having no non-trivial *clause-factors*  $F'$ , which are  $F' \subseteq F$  logically equivalent to some clause. The derived factorisations allow to reduce FC to the clause-irreducible case. Another new tool is *nearly-full-subsumption resolution* (nfs-resolution), which allows to change certain pairs  $C, D$  of clauses. Clause-sets which become clause-reducible after a series of nfs-resolutions are called *nfs-reducible*, and we can furthermore reduce FC to the nfs-irreducible case.

**Keywords:** minimal unsatisfiability , hitting clause-set , disjoint/orthogonal tautology , deficiency , Finiteness Conjecture , singular variables , full subsumption resolution , irreducible CNF , clause-factor

## 1 Introduction

Disjoint or orthogonal DNFs (every two terms/conjuncts/cubes have a conflict) have been playing an important role for boolean functions and their applications from the beginning, exploiting that the tautology problem (and also the counting problem) is computable in polynomial time; see [5, Section 1.6, Chapter 7] for some overview. As CNFs, they are more known as hitting clause-sets, denoted by

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$\mathcal{HIT}$ , and one of their earliest use is [7] (for counting solutions; see [18, Section 13.4.2] for an extension). In this report, we study the unsatisfiable elements of  $\mathcal{HIT}$ , denoted by  $\mathcal{UHIT}$ ; see [9, Section 11.4.2] for some basic information. Our main context is the study of minimally unsatisfiable clause-sets ( $\mathcal{MU}$ ; see [9]), which is organised in layers by the deficiency  $\delta$ , and where the central Finiteness Conjecture is that every such layer can be described by finitely many “patterns”. For  $\mathcal{UHIT} \subset \mathcal{MU}$  this means that every layer contains only finitely many isomorphism types (after a basic reduction), and this is the main problem studied in this report. The basic definitions are as follows.

$\mathcal{HIT}$  is the set of clause-sets  $F$ , such that for all  $C, D \in F$ ,  $C \neq D$ , there is  $x \in C$  with  $\bar{x} \in D$ . The set of unsatisfiable hitting clause-sets, denoted by  $\mathcal{UHIT}$ , is the set of  $F \in \mathcal{HIT}$  with  $\sum_{C \in F} 2^{-|C|} = 1$ . As measures we use  $c(F) := |F| \in \mathbb{N}_0$  for the number of clauses of  $F$ , and  $n(F) := |\text{var}(F)| \in \mathbb{N}_0$  for the number of variables of  $F$ , while the deficiency is defined as  $\delta(F) := c(F) - n(F) \in \mathbb{Z}$ . For  $F \in \mathcal{UHIT}$  holds  $\delta(F) \geq 1$  (an instructive exercise for the reader, or see [9]). Finally  $\mathcal{UHIT}' \subset \mathcal{UHIT}$ , the set of nonsingular  $F \in \mathcal{UHIT}$ , is given by the condition, that for every  $v \in \text{var}(F)$  there exist (at least) four different clauses  $A, B, C, D \in F$  with  $v \in A, B$  and  $\bar{v} \in C, D$ . A central problem of the field is the *Finiteness Conjecture* (FC; Conjecture 25 in [12]):

**Definition 1.**  $\text{NV}(k) \in \mathbb{N}_0 \cup \{+\infty\}$  is the supremum of  $n(F)$  for  $F \in \mathcal{UHIT}'_{\delta=k}$ .

*Conjecture 2.* For every  $k \in \mathbb{N}$  we have  $\text{NV}(k) < +\infty$ .

*Example 3.* By [6] we know  $\text{NV}(1) = 0$  (via  $\{\perp\}$ ). By [8] up to isomorphism there are two elements in  $\mathcal{UHIT}'_{\delta=2}$ :  $\mathcal{F}_2 := \{\{1, 2\}, \{-1, -2\}, \{-1, 2\}, \{-2, 1\}\}$  and  $\mathcal{F}_3 := \{\{1, 2, 3\}, \{-1, -2, -3\}, \{-1, 2\}, \{-2, 3\}, \{-3, 1\}\}$ . Thus  $\text{NV}(2) = 3$ ,

Using  $\{C\} \odot F := \{C \cup D : D \in F\}$  for clauses  $C$  and clause-sets  $F$  with  $\text{var}(C) \cap \text{var}(F) = \emptyset$ , we obtain more examples with high  $\text{NV}(k)$ :

**Lemma 4.** For  $m \in \mathbb{N}$  let  $K_m$  be defined as follows:  $K_1 := \mathcal{F}_3$ , while  $K_{m+1}$  is obtained from  $K_m$  by taking a copy  $F'$  of  $\mathcal{F}_3$  with  $\text{var}(F') \cap \text{var}(K_m) = \emptyset$ , take a new variable  $v \notin \text{var}(K_m) \cup \text{var}(F')$ , and let  $K_{m+1} := (\{\{v\}\} \odot K_m) \cup (\{\{\bar{v}\}\} \odot F')$ . Then  $K_m \in \mathcal{UHIT}'_{\delta=m+1}$  with  $n(K_m) = 3 + (m-1) \cdot 4$ . So we get  $\text{NV}(k) \geq 3 + (k-2) \cdot 4 = 4k - 5$  for  $k \geq 2$ .

We believe the  $K_m$  have the maximal number of variables for deficiency  $m+1$ , and so we consider the following strengthening of Conjecture 2:

*Conjecture 5.* For  $k \in \mathbb{N}$ ,  $k \geq 2$ , we have  $\text{NV}(k) = 4k - 5$ .

The values of  $k \mapsto 4k - 5$  for  $2 \leq k \leq 6$  are 3, 7, 11, 15, 19. The main result of this paper is that Conjecture 5 holds for  $k = 3$  (Corollary 56). New tools have been developed to show this. First we investigate singular DP-reduction [13,14], and especially its inversion called “singular extensions”, in Sections 3, 4. The main novel concept of this report is *irreducibility*, an important and intuitive concept, introduced and developed in Section 5: one can not factor out a sub-clause-set logically equivalent to a single clause. We extracted it from our work, and later

realised that up to the setting it is basically the same as investigated in [10,2]. For this report the main point is that FC can be reduced to the irreducible case via induction. This induction still leaves some leeway, and allowing “nearly-full-subsumption resolution” in Section 6 we can handle deficiency 3.

## 2 Preliminaries

Most notations and concepts in this section are standard (see the Handbook chapter [9]), but we provide all definitions, boldfacing those where confusions are possible. We use standard set-theoretical notations and concepts. For example for a set  $X$  of sets by  $\bigcup X$  the union of the elements of  $X$  is denoted, and by  $\bigcap X$  for  $X \neq \emptyset$  their intersection. The symmetric difference of sets  $X, Y$  is  $X \Delta Y := (X \setminus Y) \cup (Y \setminus X)$ . We use  $\mathbb{N} = \{x \in \mathbb{Z} : x \geq 1\}$  and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

In this report w.l.o.g. we use  $\mathcal{VA} := \mathbb{N}$  for the set of variables, that is, variables are just natural numbers, and  $\mathcal{LIT} := \mathbb{Z} \setminus \{0\}$ , that is, literals are just non-zero integers, while complementation (logical negation of literals) is just (arithmetical) negation, that is, for  $x \in \mathcal{LIT}$  we use  $\bar{x} := -x \in \mathcal{LIT}$ . For a set  $L \subseteq \mathcal{LIT}$  of literals we use  $\bar{L} := \{\bar{x} : x \in L\}$  for elementwise complementation. A **clause** is a finite set  $C \subset \mathcal{LIT}$  of literals, which is “clash-free”, that is,  $C \cap \bar{C} = \emptyset$ ; the set of all clauses is denoted by  $\mathcal{CL}$ . A **clause-set** is a finite set of clauses, the set of all clause-sets is denoted by  $\mathcal{CLS}$ . The underlying variable of a literal, given by  $\text{var} : \mathcal{LIT} \rightarrow \mathbb{N}$ , is defined as  $\text{var}(x) := |x|$  for  $x \in \mathcal{LIT}$ , while for a clause  $C$  let  $\text{var}(C) := \{\text{var}(x) : x \in C\} \subset \mathcal{VA}$ , and for a clause-set  $F$  let  $\text{var}(F) := \bigcup_{C \in F} \text{var}(C) \subset \mathcal{VA}$ . For a set  $L \subseteq \mathcal{LIT}$  of literals let  $\text{lit}(L) := L \cup \bar{L}$  be the closure under complementation, while for  $F \in \mathcal{CLS}$  let  $\text{lit}(F) := \text{lit}(\text{var}(F))$ . We note here that the actually occurring literals of  $F$  are just the elements of  $\bigcup F$ . As measures for clause-sets  $F$  we use  $n(F) := |\text{var}(F)| \in \mathbb{N}_0$  for the number of variables, and  $c(F) := |F| \in \mathbb{N}_0$  for the number of clauses. The **deficiency**  $\delta(F) \in \mathbb{Z}$  is defined as  $\delta(F) := c(F) - n(F)$ . For  $\mathcal{C} \subseteq \mathcal{CLS}$  we use notations like  $\mathcal{C}_{\delta=k} := \{F \in \mathcal{C} : \delta(F) = k\}$ . For  $F \in \mathcal{CLS}$  and  $x \in \mathcal{LIT}$  let  $\mathbf{F}_x := \{C \in F : x \in C\} \in \mathcal{CLS}$  be the sub-clause-set consisting of all clauses containing literal  $x$ , and let  $\text{ld}_F(x) := c(\mathbf{F}_x) \in \mathbb{N}_0$  be the **literal-degree** of  $x$  in  $F$ , while for  $v \in \mathcal{VA}$  the **variable-degree** is  $\text{vd}_F(v) := \text{ld}_F(v) + \text{ld}_F(\bar{v}) \in \mathbb{N}_0$ . A **full clause** of  $F \in \mathcal{CLS}$  is some  $C \in F$  with  $\text{var}(C) = \text{var}(F)$ , while the set of all full clauses over some finite  $V \subset \mathcal{VA}$  is denoted by  $\mathbf{A}(V) := \{C \in \mathcal{CL} : \text{var}(C) = V\} \in \mathcal{CLS}$ . So the set of full clauses of  $F \in \mathcal{CLS}$  is  $F \cap \mathbf{A}(\text{var}(F))$ . Furthermore we use  $A_n := \mathbf{A}(\{1, \dots, n\})$  for  $n \in \mathbb{N}_0$ . So  $A_0 = \{\perp\}$  and  $A_1 = \{\{1\}, \{-1\}\}$ . A **full variable** of  $F \in \mathcal{CLS}$  is some  $v \in \text{var}(F)$  such that for all  $C \in F$  holds  $v \in \text{var}(C)$ . So the subsets of  $\mathbf{A}(V)$  are precisely the clause-sets where every variable is full.

$\mathcal{SAT}$  is the set of satisfiable clause-sets, which are those  $F \in \mathcal{CLS}$  such that there is  $C \in \mathcal{CL}$  with  $\forall D \in F : C \cap D \neq \emptyset$ , while  $\mathcal{USAT} := \mathcal{CLS} \setminus \mathcal{SAT}$  is the set of unsatisfiable clause-sets. So  $F \in \mathcal{CLS}$  is unsatisfiable iff for all  $C \in \mathcal{CL}$  there is  $D \in F$  with  $C \cap D = \emptyset$ . Furthermore  $\mathcal{MU} \subset \mathcal{USAT}$ , the set of **minimally unsatisfiable clause-sets**, is the set of all  $F \in \mathcal{USAT}$  such that for

all  $C \in F$  holds  $F \setminus \{C\} \in \mathcal{SAT}$ . In the report we do not use the usual “partial assignments”, but just use clauses, whose elements in such a context are thought to be set to true. So in the above definition of  $\mathcal{SAT}$  the clause  $C$  corresponds to a “satisfying (partial) assignment”. This usage of clauses depends on clauses  $C$  not being tautological, i.e.,  $C \cap \overline{C} = \emptyset$  — otherwise we had an inconsistency.

$F \in \mathcal{CLS}$  is called **irredundant**, if for all  $C \in F$  there exists a super-clause  $D \in \mathcal{CL}$ ,  $C \subseteq D$ , such that for all  $E \in F \setminus \{C\}$  holds  $D \cap \overline{E} \neq \emptyset$ ; the set of all irredundant clause-sets is denoted by  $\mathcal{IRD} \subset \mathcal{CLS}$ . We note that for  $F \in \mathcal{IRD}$  and  $F' \subseteq F$  also  $F' \in \mathcal{IRD}$  holds. We have  $\mathcal{MU} \subset \mathcal{IRD}$ , and indeed  $\mathcal{MU} = \mathcal{USAT} \cap \mathcal{IRD}$ . Two clause-sets  $F, G$  are **logically equivalent** iff  $\forall C \in \mathcal{CL} : (\forall D \in F : C \cap D \neq \emptyset) \Leftrightarrow (\forall D \in G : C \cap D \neq \emptyset)$ . So  $F \in \mathcal{CLS}$  is irredundant iff there is no  $C \in F$  such that  $F$  is logically equivalent to  $F \setminus \{C\}$  iff subsets  $F', F'' \subseteq F$  are logically equivalent only if they are equal.

Two clause-sets  $F, G$  are **isomorphic**, written  $F \cong G$ , if there is a bijection (an “isomorphism”)  $f : \text{lit}(F) \rightarrow \text{lit}(G)$  with  $f(\overline{x}) = \overline{f(x)}$  for  $x \in \text{lit}(F)$  and  $G = \{\{f(x) : x \in C\} : C \in F\}$  (see “mixed symmetries” in [17, Section 10.4]).

$\mathcal{HIT}$  is the set of **hitting clause-sets**, i.e., those  $F \in \mathcal{CLS}$  such that for all  $C, D \in F$ ,  $C \neq D$ , holds  $C \cap \overline{D} \neq \emptyset$ . We have  $\mathcal{HIT} \subset \mathcal{IRD}$ . The central class for this report is  $\mathcal{UHIT} := \mathcal{HIT} \cap \mathcal{USAT}$ . Obviously  $A(V) \in \mathcal{UHIT}$ . If  $F \in \mathcal{UHIT}$  has at least two unit-clauses, then  $F \cong A_1$ . If for  $F \in \mathcal{CLS}$  holds  $\sum_{C \in F} 2^{-|C|} < 1$ , then  $F \in \mathcal{SAT}$ , while for all  $F \in \mathcal{HIT}$  holds  $\sum_{C \in F} 2^{-|C|} \leq 1$ , and for  $F \in \mathcal{HIT} \cup \mathcal{USAT}$  holds  $F \in \mathcal{UHIT} \Leftrightarrow \sum_{C \in F} 2^{-|C|} = 1$ .

The default interpretation of clause-sets  $F$  is as a CNF (conjunction of disjunction), and so the logical conjunction for  $F, G \in \mathcal{CLS}$  is just realised by  $F \cup G$ , while the logical disjunction is union clause-wise:

**Definition 6.** For clause-sets  $F, G \in \mathcal{CLS}$  we construct  $F \circledast G \in \mathcal{CLS}$ , the **combinatorial disjunction** of  $F, G$ , as the set of all clauses  $C \cup D$  for  $C \in F$  and  $D \in G$  (since clauses are clash-free, only non-clashing pairs  $C, D$  are considered here):  $F \circledast G := \{C \cup D \mid C \in F \wedge D \in G \wedge C \cap \overline{D} = \emptyset\}$ .

$F \circledast G$  is logically equivalent to the disjunction of  $F$  and  $G$ . So for  $G \in \mathcal{USAT}$  we have that  $F \circledast G$  is logically equivalent to  $F$ . And  $F \circledast G \in \mathcal{USAT} \Leftrightarrow \{F, G\} \subseteq \mathcal{USAT}$ . For a finite  $V \subset \mathcal{VA}$  we have  $A(V) = \circledast_{v \in V} \{\{v\}, \{\overline{v}\}\}$ . As  $\mathcal{USAT}$  is stable under  $\circledast$ , so is  $\mathcal{HIT}$ , and thus also  $\mathcal{UHIT}$ .

The **resolution operation**  $C \diamond D \in \mathcal{CL}$  for clauses  $C, D \in \mathcal{CL}$  is only partially defined, namely for  $|C \cap \overline{D}| = 1$ , in which case  $C \diamond D := (C \cup D) \setminus \text{lit}(C \cap \overline{D})$ , or, in other words, if there is a literal  $x$  with  $x \in C$ ,  $\overline{x} \in D$ , and  $(C \cup D) \setminus \{x, \overline{x}\}$  is a clause. **DP-reduction** is denoted for  $F \in \mathcal{CLS}$  and  $v \in \mathcal{VA}$  by  $F \rightsquigarrow \mathbf{DP}_v(F) := \{C \diamond D : C, D \in F, C \cap \overline{D} = \{v\}\} \cup \{C \in F : v \notin \text{var}(C)\}$  (also called “variable elimination”), that is, replacing all clauses containing  $v$  by their resolvents.  $\mathcal{UHIT}$  behaves well for (general) DP-reductions ([14]): it is stable, and a sequence of DP-reductions does not depend on the order.

A special case of resolution, where both parent clauses are identical up to the resolution literals, is called “full subsumption resolution”, and the corresponding resolutions and “inverse resolutions” are performed abundantly. Basic theory and applications one finds in [15, Section 6] and [16, Section 5]:

**Definition 7.** Using a slight abuse of language, a **full subsumption pair** (short “fs-pair”) is a set  $\{C, D\}$  such that  $C, D \in \mathcal{CL}$ ,  $|C \cap \overline{D}| = 1$ , and  $|C \Delta D| = 2$ . A **full subsumption resolution** (“fs-resolution”) can be performed for  $F \in \mathcal{CLS}$ , if there is an fs-pair  $\{C, D\} \subseteq F$ , such that  $C \diamond D \notin F$ , in which case  $F$  is called **full subsumption resolvable** (“fs-resolvable”), and performing the fs-resolution means the transition  $F \rightsquigarrow (F \setminus \{C, D\}) \cup \{C \diamond D\}$ . An fs-resolution is called **strict**, if no variable is lost in the transition, otherwise **non-strict**, while if we just speak of “fs-resolution”, then it may be strict or non-strict. In the other direction we speak of **(strict/non-strict) full subsumption extension** (“fs-extension”), that is, the transition  $F \in \mathcal{CLS} \rightsquigarrow F' \in \mathcal{CLS}$ , such that  $F'$  is (strict/non-strict) fs-resolvable, and the fs-resolution yields  $F$ .

In other words, for a clause  $C \in F$  and a variable  $v \in \mathcal{VA} \setminus \text{var}(C)$  we can perform an fs-extension on  $C$ , replacing  $C$  by  $C \cup \{v\}$ ,  $C \cup \{\overline{v}\}$ , iff none of these two clauses is already in  $F$  (which is guaranteed for irredundant  $F$ ); strictness means  $v \in \text{var}(F)$ , non-strictness means  $v \notin \text{var}(F)$  (i.e., the fs-extension introduces a new variable). Obviously an fs-pair  $\{C, D\}$  is logically equivalent to  $\{C \diamond D\}$ , and indeed for clauses  $C, D \in \mathcal{CL}$  there exists a clause  $E \in \mathcal{CL}$  such that  $\{C, D\}$  is logically equivalent to  $\{E\}$  iff either  $C \subseteq D$  or  $D \subseteq C$  or  $\{C, D\}$  is an fs-pair. This topic will be taken up again by the notion of a “clause-factor” (Section 5)

### 3 Singular variables

[14, Section 3] started a systematic investigation into **singular DP-reduction** (which of course played already an important role in earlier work on MU, e.g. [8]). A **singular variable** of  $F \in \mathcal{CLS}$  is a variable  $v$  with  $\min(\text{ld}_F(v), \text{ld}_F(\overline{v})) = 1$ , while a clause-set  $F \in \mathcal{CLS}$  is called **nonsingular** if  $F$  does not have singular variables; denoting the set of singular variables of  $F$  with  $\text{var}_s(F) \subseteq \text{var}(F)$ , thus  $F$  is nonsingular iff  $\text{var}_s(F) = \emptyset$ . The subsets of nonsingular elements of  $\mathcal{MU}$  and  $\mathcal{UHIT}$  are denoted by  $\mathcal{MU}'$  and  $\mathcal{UHIT}'$ . For  $F \in \mathcal{CLS}$  a singular DP-reduction is the transition  $F \rightsquigarrow \text{DP}_v(F)$  for a singular variable  $v \in \text{var}_s(F)$ . More precisely we call a variable  $v$   **$m$ -singular** for  $F$  and  $m \in \mathbb{N}$  if  $v$  is singular and  $\text{vd}_F(v) = m + 1$ ; the set of all 1-singular variables of  $F$  is denoted by  $\text{var}_{1s}(F) \subseteq \text{var}_s(F)$ , while the set of **non-1-singular variables** is  $\text{var}_{\neg 1s}(F) := \text{var}_s(F) \setminus \text{var}_{1s}(F)$ . By [14, Lemma 12, Part 2(b)] we have:

**Lemma 8 ([14]).**  $\{\bigcap F_v, \bigcap F_{\overline{v}}\}$  is an fs-pair for all  $F \in \mathcal{UHIT}$ ,  $v \in \text{var}_s(F)$ .

I.e., let  $C \in F$  be the **main clause** and  $D_1, \dots, D_m \in F$  be the **side clauses** of the  $m$ -singular variable  $v \in \text{var}_s(F)$ : Lemma 8 says  $\{C, \bigcap_{i=1}^m D_i\}$  is an fs-pair. So a 1-singular variable for UHIT is the situation of a non-strict fs-resolution:

**Corollary 9.** For  $F \in \mathcal{UHIT}$  and  $v \in \text{var}_{1s}(F)$ :  $F_v \cup F_{\overline{v}}$  is an fs-pair.

**Corollary 10.** Consider  $F \in \mathcal{UHIT}$  and a 2-singular variable  $v$ . Then the side-clauses  $D_1, D_2 \in F$  yield an fs-pair  $\{D_1, D_2\}$ .

*Proof.* Consider the main clause  $C$ , w.l.o.g. assume  $v \in C$ , and let  $C_0 := C \setminus \{v\}$ . Then  $C_0 \cup \{\bar{v}\} = D_1 \cap D_2$ . Since  $D_1, D_2$  clash, there is  $w \in \text{var}(F)$  with w.l.o.g.  $w \in D_1, \bar{w} \in D_2$ , and thus  $w \notin \text{var}(C)$ . If there would be some other literal, say w.l.o.g.  $x \in D_2 \setminus (C_0 \cup \{\bar{v}, \bar{w}\})$ , then the assignment setting all literals in  $C_0$  to false and setting  $v, w, x$  to true would satisfy  $F$  (due to  $F \in \mathcal{HIT}$ ).  $\square$

**Corollary 11.** *If  $F \in \mathcal{UHIT}$  contains a variable occurring at most three times, then this variable is a singular variable, and  $F$  contains an fs-pair.*

In Lemma 53 we give further sufficient criterion for the presence of fs-pairs.

**Corollary 12.** *For  $x, y \in C \in F \in \mathcal{UHIT}$ ,  $x \neq y$ :  $\text{ld}_F(x) = 1 \Rightarrow \text{ld}_F(y) \geq 2$ .*

*Proof.* Consider  $D \in F$  with  $\bar{x} \in D$ ; by Lemma 8  $y \in D$ , thus  $\text{ld}_F(y) \geq 2$ .  $\square$

By [14, Theorem 23] we know that singular DP-reduction is confluent for  $\mathcal{UHIT}$ . So we have the retraction  $\text{sNF} : \mathcal{UHIT} \rightarrow \mathcal{UHIT}'$ , which maps  $F \in \mathcal{UHIT}$  to the unique nonsingular  $\text{sNF}(F)$  obtainable from  $F$  by iterated singular DP-reduction.  $\mathcal{UHIT}$  is partitioned into the **singular fibres**  $\text{sNF}^{-1}(F)$  for  $F \in \mathcal{UHIT}'$ . More generally, by [14, Theorem 63] the singularity index  $\text{si}(F) \in \mathbb{N}_0$  is defined for  $F \in \mathcal{MU}$  as the unique number of singular DP-reductions needed to reduce  $F$  to an element of  $\mathcal{MU}'$ ; for  $F \in \mathcal{UHIT}$  the uniqueness of the number of reductions steps also follows with the help of the confluence of sDP-reduction. We have  $\text{si}(F) = c(F) - c(\text{sNF}(F)) = n(F) - n(\text{sNF}(F))$  for  $F \in \mathcal{UHIT}$ .

Consider  $m \in \mathbb{N}$ ; a *general  $m$ -singular extension* of  $G \in \mathcal{CLS}$  with  $x \in \mathcal{LIT} \setminus \text{lit}(F)$  is some  $F \in \mathcal{CLS}$  with  $\text{ld}_F(x) = 1$ ,  $\text{ld}_F(\bar{x}) = m$ , and  $\text{DP}_{\text{var}(x)}(F) = G$ . By [14, Lemma 9] we know that  $F \in \mathcal{MU}$  implies  $G \in \mathcal{MU}$ , and since DP-reduction is satisfiability-equivalent, we have that  $G \in \mathcal{USAT}$  implies  $F \in \mathcal{USAT}$ , however in general  $G \in \mathcal{MU}$  does not imply  $F \in \mathcal{MU}$ , since there might be tautological resolvents, and some resolvents might already exist in  $F$ . This is excluded by the definition of a “ $m$ -singular extensions” in [15, Definition 5.6], which we need to generalise in order not just to preserve  $\mathcal{MU}$ , but also  $\mathcal{UHIT}$ . Consider  $\mathcal{C} \subseteq \mathcal{CLS}$ ,  $G \in \mathcal{C}$ ,  $m \in \mathbb{N}$  and  $x \in \mathcal{LIT} \setminus \text{lit}(G)$ . A general  $m$ -singular extension  $F$  of  $G$  with  $x$  is called an  **$m$ -singular  $\mathcal{C}$ -extension of  $G$  with  $x$**  if  $F \in \mathcal{C}$  and  $c(F) = c(G) + 1$ . For “hitting extensions” we need to obey Lemma 8 and obtain:

**Lemma 13.** *For  $G \in \mathcal{UHIT}$ ,  $x \in \mathcal{LIT} \setminus \text{lit}(F)$  and  $m \in \mathbb{N}$  the  **$m$ -singular hitting extensions**  $F$  (the  $m$ -singular  $\mathcal{UHIT}$ -extensions of  $G$ ) are given by choosing some  $G' \subseteq G$  with  $c(G') = m$  such that the clause  $\bigcap G'$  clashes with every element of  $G \setminus G'$ , and letting  $F := (G \setminus G') \cup \{(\bigcap G') \cup \{x\}\} \cup \{\bar{x}\} \odot G'$ .*

Two principal choices for  $G'$  are always possible (the *trivial singular hitting extensions*): The 1-singular hitting extensions are precisely the non-strict fs-extensions. At the other end, a  $c(G)$ -singular hitting extension of  $G$  adds the unit-clause  $\{x\}$  and adds to every other clause the literal  $\bar{x}$ ; these extensions are called **full singular unit-extensions**. A simple observation:

**Lemma 14.** *Consider  $F \in \mathcal{UHIT} \setminus \{\perp\}$  and obtain  $F'$  by full singular unit-extension. Then  $F'$  has an fs-pair if and only if  $F$  has an fs-pair.*

We conclude this section by some applications to the structure of  $\mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$ , using the minimal var-degree  $\mu\text{vd}(F) := \min_{v \in \text{var}(F)} \text{vd}_F(v)$ , where by [11] for  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}_{\delta=2}$  holds  $\mu\text{vd}(F) \in \{2, 3, 4\}$ .

**Lemma 15.** *Consider  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}_{\delta=2}$  with  $\mu\text{vd}(F) = 4$ .*

1.  *$F$  is singular iff  $F$  has a unit-clause iff  $F$  is not isomorphic to  $\mathcal{F}_2$  or  $\mathcal{F}_3$ .*
2.  *$F$  is obtained from  $\mathcal{F}_2$  or  $\mathcal{F}_3$  by a series of full singular unit-extensions.*
3.  *$F$  is not fs-resolvable iff  $F$  is obtained from  $\mathcal{F}_3$  by a series of full singular unit-extensions.*

*Proof.* [15, Lemma 5.13] proves Part 1. Part 2 follows by induction, using Part 1 and the fact, that singular DP-reduction does not decrease the minimum var-degree ([15, Lemma 5.4]). Finally Part 3 follows with Lemma 14.  $\square$

**Corollary 16.**  *$F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}_{\delta=2}$  is not fs-resolvable iff  $F$  is obtained from a clause-set isomorphic to  $\mathcal{F}_3$  by a series of full singular unit-extensions (or, equivalently, unit-clause propagation on  $F$  yields a clause-set isomorphic to  $\mathcal{F}_3$ ).*

*Proof.* We have  $\mu\text{vd}(F) \in \{2, 3, 4\}$ . If  $\mu\text{vd}(F) \leq 3$ , then by Corollary 11  $F$  is fs-resolvable, while every clause-set obtained from  $\mathcal{F}_3$  by a series of full singular unit-extensions has  $\mu\text{vd}(F) \geq 4$ .  $\square$

Using that all  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}_{\delta=1}$  are fs-resolvable except of  $F = \{\perp\}$ , we get:

**Corollary 17.**  *$F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  with  $c(F) \leq 5$  is not fs-resolvable iff  $F = \{\perp\}$  or  $F \cong \mathcal{F}_3$ .*

## 4 Number of singular variables vs the singularity index

**Definition 18.** *For  $F \in \mathcal{CLS}$  let  $\mathbf{n}_s(F) := |\text{var}_s(F)| \in \mathbb{N}_0$ , while  $\mathbf{n}_{1s}(F) := |\text{var}_{1s}(F)| \in \mathbb{N}_0$  and  $\mathbf{n}_{-1s}(F) := |\text{var}_{-1s}(F)| \in \mathbb{N}_0$ .*

Thus  $n_s(F) = n_{1s}(F) + n_{-1s}(F)$ . We show that for  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  with “large”  $\text{si}(F)$  also  $n_s(F)$  must be “large” (proving [14, Conjecture 76]). First an auxiliary lemma, showing how we can reduce the number of singular variables together with the singularity index:

**Lemma 19.** *Consider  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  with  $\text{var}_s(F) \neq \emptyset$ . Then there is a singular tuple  $\mathbf{v} = (v_1, \dots, v_m)$  for  $F$  with  $1 \leq m \leq 2$  such that  $\text{var}_s(\text{DP}_{\mathbf{v}}(F)) \subseteq \text{var}_s(F) \setminus \text{var}(\{v_1, \dots, v_m\})$  (recall the order-independency of DP for  $\mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$ ). More specifically, we can choose  $\mathbf{v} = (v)$  for every  $v \in \text{var}_{-1s}(F)$ ; assume  $\text{var}_{-1s}(F) = \emptyset$  in the sequel. For  $v \in \text{var}_{1s}(F)$  there is a clause  $C$  such that  $C \cup \{v\}, C \cup \{\bar{v}\} \in F$  (Corollary 8). We can choose again  $\mathbf{v} = (v)$  if for all  $x \in C$  we have  $\text{ld}_F(x) \geq 3$ . Otherwise consider some  $x \in C$  with  $\text{ld}_F(x) = 2$  and  $\text{ld}_F(\bar{x}) \geq 2$ . Now we can choose  $\mathbf{v} = (v, \text{var}(x))$ .*

**Lemma 20.** *For  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  holds  $n_s(F) \geq \frac{1}{2} \text{si}(F)$ .*

*Proof.* We use induction on  $\text{si}(F)$ . The statement holds trivially for  $\text{si}(F) = 0$ , and so assume  $\text{si}(F) > 0$ . Consider a singular tuple  $\mathbf{v} = (v_1, \dots, v_m)$  for  $F$  according to Lemma 19, and let  $F' := \text{DP}_{\mathbf{v}}(F)$  (note that  $\text{si}(F') = \text{si}(F) - m$ ). Applying the induction hypothesis to  $F'$  we get  $n_s(F') \geq \frac{1}{2} \cdot \text{si}(F') = \frac{1}{2} \cdot (\text{si}(F) - m)$ .  $n_s(F) \geq \frac{1}{2} \cdot \text{si}(F) - 1$ , and thus  $n_s(F) \geq n_s(F') + 1 \geq \frac{1}{2} \cdot \text{si}(F)$ .  $\square$

So we get  $\text{si}(F) \leq 2 n_s(F)$  for  $F \in \mathcal{UHIT}$ . This can be refined:

**Corollary 21.** *For  $F \in \mathcal{UHIT}$  holds  $\text{si}(F) \leq 2 n_{1s}(F) + n_{\neg 1s}(F)$ .*

*Proof.* We perform first sDP-reduction (only) on the non-1-singular variables, until they all disappear, obtaining  $F' \in \mathcal{UHIT}$ . By [14, Corollary 25, Part 1], we have  $\text{var}(F) \setminus \text{var}(F') \subseteq \text{var}_s(F)$  and  $\text{var}_s(F') \subseteq \text{var}_s(F)$ . We now apply Lemma 20 to  $F'$ .  $\square$

As an application we obtain that after an fs-resolution on a nonsingular UHIT, three singular DP-reductions are sufficient to remove all singularities:

**Lemma 22.** *Consider an fs-resolvable  $F \in \mathcal{UHIT}'$ , where fs-resolution yields  $F'$  (thus  $F' \in \mathcal{UHIT}$ ). Then  $\text{si}(F') \leq 3$ .*

*Proof.* Let  $F' = (F \setminus \{C, D\}) \cup \{R\}$  with  $R := (C \cup D) \setminus \{v, \bar{v}\}$ . Assume  $\text{si}(F') \geq 4$ . Thus by Lemma 20 we have  $n_s(F') \geq 2$ . By Corollary 12 follows  $\text{var}_s(F') = \{v, w\}$ , where  $w \in \text{var}(R)$ , since only at most literal of  $R$  can have become singular in  $F'$ . But since  $F$  is nonsingular, the variable  $w$  is non-1-singular, contradicting Corollary 21.  $\square$

## 5 Reducing sub-clause-sets to clauses: “factors”

What clause-sets  $F$  are logically equivalent to clauses  $C$ ? If in some  $F \in \mathcal{CLS}$  we find some  $F' \subseteq F$  (logically) equivalent to  $C$ , then  $F$  is equivalent to  $(F \setminus F') \cup \{C\}$ . In preparation for the easy answer, note that for all  $F \in \mathcal{CLS} \setminus \{\top\}$  holds  $F = \{\bigcap F\} \odot \{D \setminus \bigcap F : D \in F\}$ .

**Lemma 23.** *For  $F \in \mathcal{CLS}$  and  $C \in \mathcal{CL}$  the following properties are equivalent:*

1.  $F$  is logically equivalent to  $\{C\}$ .
2.  $F \neq \top$ ,  $\bigcap F = C$ , and  $\{D \setminus C : D \in F\} \in \mathcal{USAT}$ .
3. There is  $G \in \mathcal{USAT}$ ,  $\text{var}(G) \cap \text{var}(C) = \emptyset$ , such that  $F = \{C\} \odot G$ .
4. There is  $G \in \mathcal{USAT}$  with  $F = \{C\} \odot G$ .

Clause-sets equivalent to clauses we call “clause-factors”:

**Definition 24.** A **clause-factor** is some  $F \in \mathcal{CLS} \setminus \{\top\}$  with  $\{C \setminus \bigcap F : C \in F\} \in \mathcal{USAT}$ . The **clause-factors of**  $F \in \mathcal{CLS}$  are the sub-clause-sets of  $F$  which are themselves clause-factors. A clause-factor  $F$  of  $F'$  is **trivial** if  $c(F) = 1$  or  $F \in \mathcal{USAT} \wedge F' = F$ , otherwise **nontrivial**. The **intersection of a clause-factor**  $F$  is  $\bigcap F \in \mathcal{CL}$ . The **residue of a clause-factor**  $F$  is  $\{C \setminus \bigcap F : C \in F\} \in \mathcal{USAT}$ ; a **residual clause-factor of**  $F \in \mathcal{CLS}$  is the residue of a clause-factor of  $F$ .



Subsets of irredundant clause-sets are irredundant again, and thus clause-factors of irredundant clause-sets are irredundant (as clause-sets):

**Lemma 25.** *Consider a residual clause-factor  $G$  of  $F \in \mathcal{CLS}$ . If  $F$  is irredundant, then  $G \in \mathcal{MU}$ . If  $F \in \mathcal{HIT}$ , then  $G \in \mathcal{UHIT}$ .*

### 5.1 Clause-factorisations

We see that a combinatorial disjunction  $F \odot G$  is the union of the clause-factors  $\{C\} \odot G$  for  $C \in F$ . If we want just to single out a single clause of  $F$  for this operation, keeping the rest of  $F$ , we do this by “pointing”  $F$ :

**Definition 26.** A **pointed clause-set** is a pair  $(F, C) \in \mathcal{CLS} \times \mathcal{CL}$  with  $C \in F$ . For a pointed clause-set  $(F, C)$  and  $G \in \mathcal{CLS}$  we define the **pointed combinatorial disjunction** (“pcd”; recall Definition 6) as

$$(F, C) \odot G := (F \setminus \{C\}) \cup (\{C\} \odot G) \in \mathcal{CLS}.$$

The simplest choice for  $F$  is  $\{C\} \odot G = (\{C\}, C) \odot G$ . The two simplest choices for  $G$  are  $(F, C) \odot \top = F \setminus \{C\}$  and  $(F, C) \odot \{\perp\} = F$ . Using the interpretation of clause-sets as CNFs,  $(F, C) \odot G$  is logically equivalent to  $(F \setminus \{C\}) \wedge (C \vee G)$ ; so if  $G$  is unsatisfiable, then  $(F, C) \odot G$  is logically equivalent to  $F$ .

**Definition 27.** A pointed combinatorial disjunction  $(F, C) \odot G$  (according to Definition 26) is called a **clause-factorisation (of  $F$  and  $G$  via  $C$ )**, if  $\text{var}(C) \cap \text{var}(G) = \emptyset$ , the union is disjoint (i.e.,  $(F \setminus \{C\}) \cap (\{C\} \odot G) = \emptyset$ ), and furthermore  $G \in \mathcal{USAT}$  holds. In a clause-factorisation  $(F, C) \odot G$  we call  $\{C\} \odot G$  **the factor**,  $G$  **the residual factor**, and  $F$  **the cofactor**. A clause-factorisation is **trivial**, if  $\{F, G\} \cap \{\{\perp\}\} \neq \emptyset$ , otherwise **nontrivial**.

The trivial clause-factorisations are  $(F, C) \odot \{\perp\} = F$  and  $(\{\perp\}, \perp) \odot G = G$  for  $F \in \mathcal{CLS}$  and  $G \in \mathcal{USAT}$ . Correspondingly, for the trivial factor  $\{C\}$  of  $F \in \mathcal{CLS}$ ,  $C \in F$ , the intersection is  $C$ , the residue is  $\{\perp\}$ , and the cofactor is  $F$ , while for the trivial factor  $G$  of  $G \in \mathcal{USAT}$  the intersection is  $\perp$  and the cofactor is  $\{\perp\}$ . Directly from the definitions we obtain the basic properties:

**Lemma 28.** *Consider  $F \in \mathcal{CLS}$  and a clause-factorisation  $F = (F_0, C) \odot G$ .*

1.  $\delta(F) = \delta(F_0) + \delta(G) - 1 + |\text{var}(F_0) \cap \text{var}(G)|$ .
2.  $\{F_0, G\} \subset \mathcal{MU} \Leftrightarrow F \in \mathcal{MU}$ .
3. If  $F \in \mathcal{MU}$ , then:
  - (a)  $1 \leq \delta(F_0) \leq \delta(F)$  and  $1 \leq \delta(G) \leq \delta(F)$ .
  - (b)  $\delta(F) = \delta(F_0)$  iff  $\text{var}(F_0) \cap \text{var}(G) = \emptyset$  and  $G \in \mathcal{MU}_{\delta=1}$ .
  - (c)  $\delta(F) = \delta(G)$  iff  $\text{var}(F_0) \cap \text{var}(G) = \emptyset$  and  $F_0 \in \mathcal{MU}_{\delta=1}$ .
  - (d) If  $F$  is nonsingular and  $F \neq \{\perp\}$ :
    - i. If  $\text{var}_{1s}(F_0) \cap \text{var}(G) = \emptyset$ , then  $\text{var}_{1s}(F_0) = \emptyset$ .
    - ii. If  $\text{var}(F_0) \cap \text{var}_s(G) = \emptyset$ , then  $G$  is nonsingular.
    - iii. If the factorisation is nontrivial:  $\delta(F_0) < \delta(F)$  and  $\delta(G) < \delta(F)$ .

The relation between clause-factorisations and -factors is now easy to see:

**Lemma 29.**  $F \in \mathcal{CLS}$  allows a nontrivial clause-factorisation iff  $F$  contains a nontrivial clause-factor.

Following [10,2] (introducing “irreducibility” for covers of the integers resp. cell partitions of lattice parallelotops), we introduce the fundamental notion of “clause-irreducible clause-sets”, not allowing non-trivial clause-factorisations:

**Definition 30.** A clause-set  $F \in \mathcal{CLS}$  is called **clause-irreducible**, if every clause-factor is trivial, otherwise  $F$  is called **clause-reducible**; the set of all clause-irreducible clause-sets is denoted by  $\mathbf{CIR} \subset \mathcal{CLS}$ .

## 5.2 Clause-factors for UHIT

**Definition 31.** In this report we are especially concerned with  $\mathcal{UHIT}$ , and we call the subset given by the clause-irreducible elements  $\mathbf{IUH} := \mathbf{CIR} \cap \mathcal{UHIT}$ .

So  $\mathbf{IUH}_{n \leq 1} = \mathcal{UHIT}_{n \leq 1} = \{\{\perp\}\} \cup \{\{\{v\}, \{\bar{v}\}\} : v \in \mathcal{VA}\}$ .

*Example 32.*  $\mathcal{F}_2 = \{\{1, 2\}, \{-1, -2\}, \{-1, 2\}, \{-2, 1\}\}$  is clause-reducible, and the nontrivial clause-factors are the  $\binom{4}{2} - 2 = 4$  2-element subsets of  $F$  where the two clauses have precisely one clash (these are the fs-pairs). So  $\mathbf{IUH}_{n=2} = \emptyset$ .

**Lemma 33.** Consider  $F \in \mathcal{UHIT}$  and a non-empty subset  $\top \neq F' \subseteq F$ , and let  $F'' := (F \setminus F') \cup \{\bigcap F'\}$  be the “cofactor”. We note that this union is disjoint, since  $F \in \mathcal{HIT}$ . The following conditions are equivalent:

1.  $F'$  is a clause-factor of  $F$ .
2. The intersection of  $F'$  clashes with every other clause, i.e.,  $F'' \in \mathcal{HIT}$ .
3.  $F'' \in \mathcal{UHIT}$ .

*Proof.* Part 1 implies Part 3: If  $F'$  is a factor of  $F$ , then  $F''$  is unsatisfiable, since  $\bigcap F'$  subsumes all clauses of  $F'$ , and  $F''$  is a hitting clause-set, since if there would be some  $C \in F \setminus F'$  without a clash with  $\bigcap F'$ , then setting all literals in  $C$  to false would be a satisfying assignment for the residue. Trivially Part 3 implies 2. Finally assume  $F'' \in \mathcal{HIT}$ , but that the residue  $\{C \setminus \bigcap F' : C \in F'\} \in \mathcal{SAT}$ . So then there is a clause  $D$  with  $\text{var}(D) \cap \text{var}(\bigcap F') = \emptyset$ , which has a clash with every clause in  $F'$ , and so  $D \cup \bigcap F'$  is a clause with a clash with every clause of  $F$ , contradicting unsatisfiability of  $F$ .  $\square$

Factors of UHITs are basically the same as singular extensions resulting in unsatisfiable hitting clause-sets (up to the choice of the extension-variable):

**Lemma 34.** Consider  $F \in \mathcal{UHIT}$ . Then up to the choice of the extension variable, the singular hitting extensions of  $F$  are given according to Lemma 13 by some nonempty  $G \subseteq F$ , and by Lemma 33 these subsets are precisely the factors  $F'$  of  $F$ . So the singular  $m$ -hitting-extensions for  $m \geq 1$  correspond 1-1 to the factors  $F'$  of  $F$  with  $c(F') = m$ . Especially, the trivial factors of  $F$  correspond 1-1 to the trivial singular hitting extensions of  $F$ , namely the factors of size 1 correspond to the 1-extensions, and the factors of size  $c(F)$  correspond to the full singular unit-extensions.

**Corollary 35.** *A clause-set  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  is irreducible if and only if every singular hitting-extension is trivial.*

Singular variables or full variables yield factors as follows:

**Lemma 36.** *Consider  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  and  $v \in \text{var}(F)$ . If  $v$  is a singular variable or a full variable of  $F$ , then  $F_v$ ,  $F_{\bar{v}}$  and  $F_v \cup F_{\bar{v}}$  are factors of  $F$ .*

*Proof.* First consider that  $v$  is a singular variable of  $F$ , and assume w.l.o.g. that  $\text{ld}_F(v) = 1$ . Then trivially  $F_v$  is a factor of  $F$ . Let  $C$  be the main clause of  $v$ , where w.l.o.g.  $v \in C$ , and let  $D := C \setminus \{v\}$ . Now  $\bigcap F_{\bar{v}} = D \cup \{\bar{v}\}$  (since  $F$  is hitting), and thus  $F_{\bar{v}}$  is a factor of  $F$  (since  $C$  clashes with every other clause). Finally  $\bigcap (F_v \cup F_{\bar{v}}) = D$ , and thus also  $F_v \cup F_{\bar{v}}$  is a factor. Now assume that  $v$  is a full variable of  $F$ . Then  $F_v \cup F_{\bar{v}} = F$ , while  $\bigcap F_v = \{v\}$  (and  $\bigcap F_{\bar{v}} = \{\bar{v}\}$ ; otherwise  $F$  would be satisfiable), and thus also  $F_v, F_{\bar{v}}$  are factors.  $\square$

There are two other classes of easily recognisable factors:

**Lemma 37.** *Consider  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$ . The factors  $F'$  with  $c(F') = 2$  are precisely the fs-pairs (recall Definition 7) contained in  $F$ .*

*Proof.* First consider an fs-pair  $F' := \{C \cup \{v\}, C \cup \{\bar{v}\}\} \subseteq F$ ; note that  $\bigcap F' = C$ . If there would be  $D \in F \setminus F'$  with  $C \cap \bar{D} = \emptyset$ , then  $D$  would also be clash-free with one element of  $F'$ , contradicting the hitting condition.

Now consider a factor  $F'$  with  $c(F') = 2$ , and let  $C := \bigcap F'$ . Then  $(F \setminus F') \cup \{C\} \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  by Lemma 33. Due to  $\sum_{C \in F} 2^{-|C|} = 1$  we have that  $|D| = |C| + 1$  for  $D \in F$ , and because of the hitting condition thus  $F'$  must be an fs-pair.  $\square$

**Lemma 38.** *Consider  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$ . Then the factors  $F'$  with  $c(F') = c(F) - 1$  are precisely given by  $F' = F \setminus \{C\}$  for  $C \in F$  with  $|C| = 1$  (unit-clauses).*

*So if  $c(F) = 2$  (i.e.,  $F = \{\{v\}, \{\bar{v}\}\}$  for some  $v \in \mathcal{V}A$ ), then there are precisely two such factors, while otherwise there can be at most one such factor.*

*Proof.* Consider a factor  $F'$  of  $F$  with  $c(F') = c(F) - 1$ , and let  $C := \bigcap F'$ . Since  $F' \neq F$ , we have  $|C| \geq 1$ . If  $|C| \geq 2$ , then  $F$  would be satisfiable (note that  $C$  clashes with the clause in  $F \setminus F'$ ). So there is a literal  $x$  with  $C = \{x\}$ . Now the clause of  $F \setminus F'$  must be  $\{\bar{x}\}$ , since otherwise again  $F$  would be satisfiable. The remaining assertions follow easily.  $\square$

By Lemmas 36, 37, 38:

**Lemma 39.** *A clause-irreducible  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  is singular iff it has a full variable iff it has an fs-pair iff it has a unit-clause iff  $F \cong A_1$ .*

Since every  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}_{\delta=1}$  with  $n(F) > 0$  is fs-resolvable,  $\mathcal{U}\mathcal{H}_{\delta=1} = \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}_{n \leq 1}$ . Having no fs-pair resp. no full variables has further consequences:

**Lemma 40.** *If  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  has no fs-pair, then  $F$  has no nontrivial clause-factor  $F'$  with  $c(F') \leq 4$ , while if  $F$  has no full variable, then  $F$  has no nontrivial clause-factor  $F'$  with  $c(F') \geq c(F) - 3$ .*

*Proof.* If there would be a nontrivial clause-factor  $F'$  with  $c(F') \leq 4$ , then by Corollary 17 the residue would have an fs-pair, and then also  $F$  had one. And if there would be  $F'$  with  $c(F') \geq c(F) - 3$ , then the cofactor would be an element of  $\mathcal{UHT}$  with at most four clauses, and thus had a full variable.  $\square$

**Lemma 41.** *Up to isomorphism there is one element in  $\mathcal{UHT}_{\delta=2}$ , namely  $\mathcal{F}_3$ .*

*Proof.* Up to isomorphism there are precisely two elements in  $\mathcal{UHT}'_{\delta=2}$ , namely  $\mathcal{F}_2$ , which is clause-reducible (Example 32), and  $\mathcal{F}_3$ , which has no fs-pair, and thus by Lemma 40 is clause-irreducible.  $\square$

### 5.3 Reducing FC to the irreducible case

Strengthening Lemma 28 (again with simple proofs):

**Lemma 42.** *Consider  $F \in \mathcal{CLS}$  and a clause-factorisation  $F = (F_0, C) \otimes G$ .*

1.  $\{F_0, G\} \subset \mathcal{UHT} \Leftrightarrow F \in \mathcal{UHT}$ .
2. Assume  $F \in \mathcal{UHT}'$ .
  - (a) If  $F$  is not strictly fs-resolvable (recall Definition 7), then  $\text{var}_{1s}(F_0) = \text{var}_{1s}(G) = \emptyset$ .
  - (b)  $n_s(F_0) \leq |\text{var}_s(F_0) \cap \text{var}(G)| + 1 \leq |\text{var}(F_0) \cap \text{var}(G)| + 1 \leq \delta(F)$ .
  - (c)  $n_s(G) \leq |\text{var}(F_0) \cap \text{var}_s(G)| \leq |\text{var}(F_0) \cap \text{var}(G)| \leq \delta(F) - 1$ .

**Theorem 43.** *Consider  $F \in \mathcal{UHT}' \setminus \mathcal{UHT}$  and a nontrivial clause-factorisation  $F = (F_0, C) \otimes G$ . Let  $k := \delta(F)$  (and so  $k \geq 2$ ).*

1. If  $F$  is strictly fs-resolvable, then  $n(F) \leq \text{NV}(k-1) + 3$ .
2. Otherwise  $n(F) \leq \text{NV}(\delta(F_0)) + \text{NV}(\delta(G)) + |\text{var}(F_0) \cap \text{var}(G)| + 1$ .
3. Assume that  $\forall k' \geq 2 : k' < k \Rightarrow \text{NV}(k') = 4k' - 5$ .
  - (a) If  $F$  is strictly fs-resolvable, then  $n(F) \leq 4k - 6$ .
  - (b) Otherwise  $n(F) \leq 4k - 5 - 3 \cdot |\text{var}(F_0) \cap \text{var}(G)| \leq 4k - 5$ .

*Proof.* Part 1: Perform a strict fs-resolution for  $F$ , obtaining  $F'$  (with  $\delta(F') = \delta(F) - 1$ ); by Lemma 22 we get  $n(F) = n(F') \leq n(\text{sNF}(F')) + 3 \leq \text{NV}(\delta(F) - 1) + 3$ . Part 2: Assume that  $F$  is not strictly fs-resolvable. So by Lemma 42, Part 2a, we get  $\text{var}_{1s}(F_0) = \text{var}_{1s}(G) = \emptyset$ . Let  $s := |\text{var}(F_0) \cap \text{var}(G)|$ . We have

$$\begin{aligned} n(F) &= n(F_0) + n(G) - s = (n(\text{sNF}(F_0)) + \text{si}(F_0)) + (n(\text{sNF}(G)) + \text{si}(G)) - s \leq \\ &\quad (\text{NV}(\delta(F_0)) + \text{si}(F_0)) + (\text{NV}(\delta(G)) + \text{si}(G)) - s. \end{aligned}$$

By Corollary 21 holds  $\text{si}(F_0) \leq n_s(F_0)$  and  $\text{si}(G) \leq n_s(G)$ , where by Lemma 42, Parts 2b, 2c, we have  $n_s(F_0) \leq s + 1$  and  $n_s(G) \leq s$ , which completes the proof.

Part 3a follows from Part 1 for  $k \geq 3$ :  $n(F) \leq \text{NV}(k-1) + 3 \leq 4(k-1) - 5 + 3 = 4k - 6$ . And for  $k = 2$  we get  $F \cong \mathcal{F}_2$ , since  $\mathcal{F}_3$  is not fs-resolvable, and thus  $2 = n(F) = 4k - 6$ . For Part 3b we notice that now  $k \geq 3$  holds, since  $\mathcal{F}_3$  is irreducible by Lemma 41. Lemma 28, Part 1 yields  $k = \delta(F_0) + \delta(G) + s - 1$ , where by Lemma 28, Part 3(d)iii:  $1 \leq \delta(F_0) \leq k - 1$  and  $1 \leq \delta(G) \leq k - 1$ . By Part 2 we know  $n(F) \leq \text{NV}(\delta(F_0)) + \text{NV}(\delta(G)) + s + 1$ . And by Lemma 42, Part 2a, we get  $\text{var}_{1s}(F_0) = \text{var}_{1s}(G) = \emptyset$ , and thus  $\delta(F_0), \delta(G) \geq 2$ . Now  $\text{NV}(\delta(F_0)) + \text{NV}(\delta(G)) + s + 1 = 4(k - s + 1) - 2 \cdot 5 + s + 1 = 4k - 5 - 3s$ .  $\square$

**Corollary 44.** *Conjecture 2 is equivalent to the statement, that for all  $k \geq 3$  we have  $\sup\{n(F) : F \in \mathcal{I}\mathcal{U}\mathcal{H}_{\delta=k}\} < +\infty$ . And Conjecture 5 is equivalent to the statement, that for all  $k \geq 3$  we have  $\sup\{n(F) : F \in \mathcal{I}\mathcal{U}\mathcal{H}_{\delta=k}\} \leq 4k - 5$ .*

In Corollary 52 we will further restrict the critical cases.

## 6 Subsumption-flips

Recall that  $C, D \in \mathcal{CL}$  are *full-subsumption resolvable* (“fs-resolvable”; Definition 7) iff  $C, D$  are resolvable and  $|C \Delta D| = 2$ . In the following we write at places  $A \cup B := A \cup B$  in case  $A \cap B = \emptyset$ .

**Definition 45.** *Clauses  $C, D$  are **nearly-full-subsumption resolvable** (nfs-resolvable), and  $\{C, D\}$  is an **nfs-pair**, if  $C, D$  are resolvable and  $|C \Delta D| = 3$ .*

$C, D$  are nfs-resolvable iff there is  $E \in \mathcal{CL}$  and  $x, y \in \mathcal{LIT}$ ,  $\text{var}(x) \neq \text{var}(y)$ , with  $\{C, D\} = \{E \cup \{x\}, E \cup \{\bar{x}, y\}\}$ ; we call  $x$  the *resolution literal*,  $y$  the *side literal*, and  $E$  the *common part*.

**Definition 46.** *For an nfs-pair  $\{C, D\} = \{E \cup \{x\}, E \cup \{\bar{x}, y\}\}$ , the **nfs-flip** is the unordered pair  $\{E \cup \{x, \bar{y}\}, E \cup \{y\}\}$  (in the clause with the side literal remove the resolution literal, and for the other clause add the complemented side literal). An  $F \in \mathcal{CLS}$  is called **nfs-resolvable**, if there is an nfs-pair  $\{C, D\} \subseteq F$ , while none of the two clauses of the nfs-flip is in  $F$ . For nfs-resolvable  $F$  **on**  $\{C, D\} \subseteq F$ , the nfs-flip replaces these two clauses by the result of the nfs-flip (so the number of clauses and the set of variables stays unaltered).*

If  $\{C, D\} = \{E \cup \{x\}, E \cup \{\bar{x}, y\}\}$  are nfs-resolvable, then the result  $\{C', D'\}$  of the nfs-flip is again nfs-resolvable, and the nfs-flip yields back  $\{C, D\}$ . We can simulate the nfs-flip as follows. Performing one strict fs-extension, we obtain  $\{E \cup \{x, y\}, E \cup \{x, \bar{y}\}, E \cup \{\bar{x}, y\}\}$ . Now precisely two strict fs-resolutions are possible, yielding either the original  $\{C, D\}$  or the nfs-flip  $\{E \cup \{x, \bar{y}\}, E \cup \{y\}\}$ . So, if  $F \in \mathcal{U}\mathcal{HIT}$  contains an nfs-pair  $\{C, D\} \subseteq F$  and we replace the pair by its nfs-flip, then we obtain  $F' \in \mathcal{U}\mathcal{HIT}$ , which we say is obtained by *one nfs-flip* from  $F$ . An nfs-pair  $\{C, D\}$  and its flip  $\{C', D'\}$  are logically equivalent. Nfs-flips for  $F \in \mathcal{CLS}$  leave the measures  $n, c, \ell, \delta$  invariant, also the distribution of clause-sizes, while changing precisely the variable degree of two variables of  $F$ , one goes up and one goes down by one.

*Example 47.*  $\mathcal{F}_3 = \{\{1, 2, 3\}, \{-1, -2, -3\}, \{-1, 2\}, \{-2, 3\}, \{-3, 1\}\} \in \mathcal{I}\mathcal{U}\mathcal{H}_{\delta=2}$  is nfs-resolvable, for example the nfs-flip on the first and the third clause in  $\mathcal{F}_3$  is  $F := \{\{2, 3\}, \{-1, -2, -3\}, \{-1, 2, -3\}, \{-2, 3\}, \{-3, 1\}\}$ . Now  $F$  has several nontrivial clause-factors, namely there are two strict fs-pairs, where fs-resolution yields elements of  $\mathcal{U}\mathcal{HIT}_{\delta=1}$ , and 1 is a 2-singular variable of  $F$  (with  $\text{SNF}(F) \cong \mathcal{F}_2$ ), yielding one further nontrivial clause-factor.

**Definition 48.**  $F \in \mathcal{IUH}$  is **nfs-reducible**, if via a series of nfs-flips  $F$  can be transformed into a clause-reducible clause-set, otherwise  $F$  is **nfs-irreducible**; the set of all nfs-irreducible elements of  $\mathcal{IUH}$  is denoted by  $\mathcal{NIUH} \subset \mathcal{IUH}$ .

By Lemma 41 and Example 47:

**Lemma 49.**  $\mathcal{NIUH}_{\delta=2} = \emptyset$ .

If after an nfs-flip we obtain non-singularity, then it is of the easiest form, and after re-singularisation we have clause-reducibility with additional properties:

**Lemma 50.** Consider  $F \in \mathcal{UHT}'$  with an nfs-flip  $F'$ . Then  $\text{si}(F') \leq 1$ . Assume that  $F$  not fs-resolvable and  $\text{si}(F') = 1$ , and let  $G := \text{sNF}(F')$ . There is a non-trivial clause-factorisation  $G = (G_0, C) \odot H$ , such that  $\text{var}(G_0) \cap \text{var}(H) \neq \emptyset$ .

*Proof.* Consider an nfs-pair  $\{E \cup \{x\}, E \cup \{\bar{x}, y\}\} \subseteq F$ , and thus  $E \cup \{x, \bar{y}\}, E \cup \{y\} \in F'$ ; we have  $\text{ld}_{F'}(\bar{x}) = \text{ld}_F(\bar{x}) - 1$ ,  $\text{ld}_{F'}(\bar{y}) = \text{ld}_F(\bar{y}) + 1$ , while all other literal degrees remain the same. So the only possibility for a singularity in  $F'$  is that  $\text{ld}_{F'}(\bar{x}) = 1$ , and then  $\text{var}_s(F') = \text{var}_{-1s}(F') = \{\text{var}(x)\}$ , whence in general  $\text{si}(F') \leq 1$ , proving the first assertion. Now consider the remaining assertions.

Consider the (single)  $\bar{x}$ -occurrence in  $F'$  (which has been transferred unchanged from  $F$ ). By Lemma 8 and the necessity of a clash with the second  $\bar{x}$ -occurrence in  $F$ , this clause is  $E' \cup \{\bar{x}, \bar{y}\} \in F' \cap F$ , where due to  $F$  not being fs-resolvable we have  $E' \subset E$ . Consider the nontrivial factor  $F'_x$  of  $F'$  according to Lemma 36: We have  $E \cup \{x, \bar{y}\} \in F'_x$ , while by Lemma 8 the intersection of  $F'_x$  is  $E' \cup \{x, \bar{y}\}$ . Note that  $E \cup \{y\} \in F' \setminus F'_x$ .

Obtain  $F''_x$  from  $F'_x$  by removing all occurrences of  $x$ . Now  $G = \text{DP}_{\text{var}(x)}(F')$  is obtained from  $F'$  by removing the clause  $E' \cup \{\bar{x}, \bar{y}\}$ , and replacing  $F'_x$  by  $F''_x$ .  $F''_x$  is a nontrivial factor of  $G$ , and via Lemma 34 we obtain the sought nontrivial clause-factorisation:  $C := E' \cup \{\bar{y}\}$ ,  $H := \{D \setminus C : D \in F''_x\}$  and  $G_0 := G \setminus F''_x$ . Due to  $E \setminus E' \in H$  and  $E \cup \{y\} \in G_0$  there is a common variable.  $\square$

**Theorem 51.** Consider  $k \geq 3$ , and assume that  $\forall k' \in \mathbb{N}_{\geq 3} : k' < k \Rightarrow \text{NV}(k') = 4k - 5$ . Then  $\text{NV}(k) = 4k - 5$  is equivalent to the statement, that for all  $F \in \mathcal{NIUH}_{\delta=k}$  holds  $n(F) \leq 4k - 5$ .

*Proof.* Clearly the condition is necessary, and it remains to show that it is sufficient. By Theorem 43, Part 3, it remains to consider  $F_0 \in \mathcal{IUH}_{\delta=k}$  and to show  $n(F_0) \leq 4k - 5$ , and by the condition we can assume that  $F_0$  is nfs-reducible. So there exists a series  $F_0, \dots, F_m$ ,  $m \geq 1$ , such that  $F_{i+1}$  is an nfs-flip of  $F_i$ , and where  $F_m$  is clause-reducible, while  $F_{m-1}$  is clause-irreducible. If  $F_m$  is non-singular, then we are done (as before), and so assume that  $F_m$  is singular. We apply Lemma 50, with  $F := F_{m-1}$  and  $F' := F_m$ , while  $G = \text{sNF}(F_m)$  with  $n(G) = n(F_0) - 1$ . If  $G$  is strictly fs-resolvable, then by Theorem 43, Part 3a we get  $n(G) \leq 4k - 6$ , so assume that  $G$  is not strictly fs-resolvable. Now by Theorem 43, Part 3b we get  $n(G) \leq 4k - 5 - 3 \cdot 1 = 4k - 8$ .  $\square$

**Corollary 52.** Conjecture 2 is equivalent to the statement, that for all  $k \geq 3$  we have  $\sup\{n(F) : F \in \mathcal{NIUH}_{\delta=k}\} < +\infty$ . And Conjecture 5 is equivalent to the statement, that for all  $k \geq 3$  we have  $\sup\{n(F) : F \in \mathcal{NIUH}_{\delta=k}\} \leq 4k - 5$ .

Before we can finally prove the main result of this report, we need two lemmas on nfs-reducibility. First we show that clause-irreducible clause-sets with a variable occurring positively and negatively exactly twice are nfs-reducible:

**Lemma 53.** *Consider  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  with  $v \in \text{var}(F)$  such that  $\text{ld}_F(v) = \text{ld}_F(\bar{v}) = 2$ . Then  $F$  is fs-resolvable or allows an nfs-flip enabling an fs-resolution.*

*Proof.* Assume that  $F$  has no fs-pairs. Let  $C_1, C_2$  be the two  $v$ -occurrences and let  $D_1, D_2$  be the two  $\bar{v}$ -occurrences. There is a literal  $x \in C_1$  with  $\bar{x} \in C_2$ . If  $\text{var}(x) \notin \text{var}(D_1) \cup \text{var}(D_2)$ , then setting  $x$  to true (or false, it doesn't matter) we create an UHIT where  $v$  becomes singular, and via Corollary 10 then  $\{D_1, D_2\}$  is an fs-pair; thus  $\text{var}(x) \in \text{var}(D_1) \cup \text{var}(D_2)$ . If  $\text{var}(x) \in \text{var}(D_1) \cap \text{var}(D_2)$ , then w.l.o.g.  $x \in D_1, \bar{x} \in D_2$  (if there wouldn't be a clash, then via setting  $x$  to true resp. false one could create an UHIT with  $v$  occurring only once); now setting  $x$  to false (or true, again it doesn't matter)  $v$  becomes 1-singular, and via Corollary 9  $\{C_1 \setminus \{x\}, D_1 \setminus \{x\}\}$  is an fs-pair, whence  $\{C_1, D_1\}$  would be an fs-pair; thus  $\text{var}(x) \notin \text{var}(D_1) \cap \text{var}(D_2)$ . So finally w.l.o.g.  $x \in D_1, \text{var}(x) \notin \text{var}(D_2)$ . So after setting  $x$  to true/false we can apply Corollary 9 resp. 10, and we obtain that there is a clause  $A$  and a new literal  $z$  such that  $C_2 = \{v, \bar{x}\} \cup A \cup \{z\}$ ,  $D_2 = \{\bar{v}\} \cup A \cup \{z\}$ , and  $D_1 = \{\bar{v}, x\} \cup A \cup \{\bar{z}\}$ . Applying the nfs-flip to  $C_2, D_2$ , from  $D_2$  we obtain  $D'_2 := \{\bar{v}, x\} \cup A \cup \{z\}$ , and now  $\{D_1, D'_2\}$  is an fs-pair.  $\square$

Furthermore, if we can reach deficiency 1 by assigning one variable, then via a series of nfs-flips we can create an fs-pair:

**Lemma 54.** *Consider  $F \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  and  $x \in \text{lit}(F)$  such that assigning  $x$  to true in  $F$  yields a clause-set with deficiency 1. Then via a series of nfs-flips on  $F$  we can reach an element of  $\mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}$  with an fs-pair.*

*Proof.* If  $F$  has an fs-pair, we are done, and so assume that  $F$  is not fs-resolvable. Let  $F'$  be the clause-set obtained by assigning  $x$  to true (so  $F' \in \mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}_{\delta=1}$ ); we do induction on  $c(F')$ . If  $c(F') = 1$ , then  $F = \{\{x\}, \{\bar{x}\}\}$ , and we are done, so assume  $c(F') > 1$ . Now  $F'$  contains an fs-pair  $\{\{y\} \cup C, \{\bar{y}\} \cup C\}$  (as was already shown in [1]), and thus  $F$  contains w.l.o.g. the nfs-pair  $\{\{y\} \cup C \cup \{\bar{x}\}, \{\bar{y}\} \cup C\}$ . Performing the nfs-flip replaces these two clauses by  $C \cup \{\bar{x}\}, \{\bar{y}\} \cup C \cup \{x\}$ , and so the new  $F'$  has been reduced by one clause (while still in  $\mathcal{U}\mathcal{H}\mathcal{I}\mathcal{T}_{\delta=1}$ ).  $\square$

**Theorem 55.**  $\mathcal{N}\mathcal{I}\mathcal{U}\mathcal{H}_{\delta=3} = \emptyset$ .

*Proof.* Consider  $F \in \mathcal{I}\mathcal{U}\mathcal{H}_{\delta=3}$ , and assume that  $F$  is nfs-irreducible. Consider some  $v \in \text{var}(F)$  with minimal  $\text{vd}_F(v)$ . So  $\text{vd}_F(v) \in \{4, 5\}$  (using Corollary 11 and [12, Theorem 15]), and by Lemma 53 we have indeed  $\text{vd}_F(v) = 5$ . W.l.o.g.  $\text{ld}_F(v) = 3$ , contradicting Lemma 54 with  $x = v$ .  $\square$

By Theorem 51:

**Corollary 56.**  $\text{NV}(3) = 4 \cdot 3 - 5 = 7$ .

## 7 Conclusion and outlook

We proved the strong form of the Finiteness Conjecture (FC, Conjecture 5) for deficiency  $k = 3$  (Corollary 56), and developed on the way new tools for understanding (hitting) clause-sets:

- Full subsumption (fs) resolution and full subsumption (fs) extension (Definition 7): new aspects of one of the oldest methods in propositional logic (at least since [4]).
- Singular variables and the singularity index (Sections 3, 4): simple variables and their elimination and introduction.
- *Clause-factors, clause-factorisations, irreducible clause-sets* (Section 5): generalising fs-resolution and singular variables through a structural approach.
- *Nearly full subsumption (nfs) resolution, nfs-irreducible clause-sets* (Section 6): extending the reach of clause-factorisations.

The proof of Corollary 56 works by the general reduction to the nfs-irreducible case (Theorem 51), where there are no such cases for deficiencies up to 3 (Theorem 55). Future steps are the determination of  $\mathcal{UHT}'_{\delta=3}$  and the proof of FC for  $k = 4$ . We believe clause-irreducible clause-sets are a valuable tool, and a fundamental question here is about a kind of prime-factorisation of UHITs into clause-irreducible clause-sets.

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